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Notes on Non-interpolation Spaces

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Some general examples of non-interpolation pairs and spaces are presented. Necessary conditions for interpolation are established which determine the first type of examples. Constructions connected with the relative completion and a property of the K-functional provide the second class of examples. These techniques provide new information about non-interpolation of symmetric spaces. © 1989 Academic Press, Inc.

1. INTRODUCTION

We recall some notation from interpolation theory (cf. [2, 9]).

A pair $\bar{A} = (A_0, A_1)$ of Banach spaces is called a *Banach couple* if A_0 and A_1 are both continuously imbedded in some Hausdorff topological vector space V.

For a Banach couple $\bar{A} = (A_0, A_1)$ we can form the *intersection* $\Delta(\bar{A}) = A_0 \cap A_1$ and the *sum* $\sum(\bar{A}) = A_0 + A_1$. They are both Banach spaces in the natural norms

$$\|a\|_{A_0 \cap A_1} = J(1, a; A_0, A_1) \qquad \text{and} \qquad \|a\|_{A_0 + A_1} = K(1, a; A_0, A_1),$$

where, for $t > 0$,

$$J(t, a) = J(t, a; A_0, A_1) = \max(\|a\|_{A_0}, t\,\|a\|_{A_1}),$$

and

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\}.$$

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A Banach space A is called an *intermediate space* between A_0 and A_1 (or with respect to $\bar{A}$) if $A_0 \cap A_1 \subset A \subset A_0 + A_1$ with continuous inclusions. For brevity, the closure of $A_0 \cap A_1$ in A will be denoted by A^0.

Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two Banach couples. We denote by $L(\bar{A}, \bar{B})$ the Banach space of all linear operators $T: A_0 + A_1 \to B_0 + B_1$ such that the restriction of T to the space A_i is a bounded operator from A_i into B_i, $i = 0, 1$, with the norm

$$\| T \|_{L(\bar{A}, \bar{B})} = \max (\| T \|_{A_0 \to B_0}, \| T \|_{A_1 \to B_1}).$$

We say that two intermediate spaces A and B are called *interpolation spaces with respect to* $\bar{A}$ *and* $\bar{B}$ and we will write $(A, B) \in \mathrm{Int}(\bar{A}, \bar{B})$ if every linear operator from $L(\bar{A}, \bar{B})$ maps A into B. It is a consequence of the closed graph theorem that then the restriction of T to A is the bounded operator from A into B and

$$\| T \|_{A \to B} \leqslant C \, \| T \|_{L(\bar{A}, \bar{B})} \tag{1}$$

for some positive constant C independent of $T \in L(\bar{A}, \bar{B})$. If A coincides with B then A is called an *interpolation space* with respect to $\bar{A}$ and $\bar{B}$ and we write $A \in \mathrm{Int}(\bar{A}, \bar{B})$; if, moreover, $A_0 = B_0$ and $A_1 = B_1$ then A is called an *interpolation space between* A_0 *and* A_1 (or with respect to $\bar{A}$), and we write $A \in \mathrm{Int}\, \bar{A}$.

Let $\mathscr{P}$ denote the set of all functions $\varphi: (0, \infty) \to (0, \infty)$ such that $\varphi(s) \leqslant \max(1, s/t)\, \varphi(t)$ for all $s, t > 0$. We then define the space $\bar{A}_{\varphi, \infty} = (A_0, A_1)_{\varphi, \infty}$ as the space of all $a \in A_0 + A_1$ such that

$$\| a \|_{\varphi, \infty} = \sup_{t > 0} \frac{K(t, a; A_0, A_1)}{\varphi(t)}$$

is finite; if $\varphi(t) = t^\theta (0 \leqslant \theta \leqslant 1)$ we write, in short, $\bar{A}_{\theta, \infty}$ and $\| a \|_{\theta, \infty}$. We note that $\bar{A}_{0, \infty}$ is the space of all $a \in A_0 + A_1$ such that $\lim_{t \to \infty} K(t, a; A_0, A_1) < \infty$; it can be proved that $\bar{A}_{0, \infty}$ is a relative completion $\tilde{A}_0$ of A_0 with respect to $A_0 + A_1$.

The plan of the paper is as follows:

In Section 2 we discuss necessary conditions for interpolation using, among other things, the fundamental function μ.

In Section 3 first we study interpolation spaces A and B with respect to $\bar{A}$ and $\bar{B}$, where A is the sum $A_0 + A_1$(Th. 1). In Theorem 2 we give a result on non-interpolation of A_i and B_{1-i} $(i = 0, 1)$ with respect to $\bar{A}$ and $\bar{B}$ based on considerations in [12]. In Theorem 3 we investigate when A_0 or A_1 is an interpolation space with respect to $\bar{A} = (A_0, A_1)$ and $\bar{B} = (A_1, A_0)$. These results contain some results of Aronszajn and Gagliardo [1].

In Section 4, the above results are applied to an important class of symmetric spaces, in particular to Lorentz spaces. For example, Theorem 4 characterizes interpolation spaces between L_∞ and E_1.

Finally, in Section 5, we have collected various results giving μ for symmetric spaces.

2. Fundamental Functions and Necessary Conditions for Interpolation

For a Banach space A containing $A_0 \cap A_1 \neq \{0\}$ (or for a Banach space A contained in $A_0 + A_1$) the *fundamental function* $\mu_A(\nu_A)$ is given for $t > 0$ by

$$\mu_A(t) = \mu_A(t, A_0, A_1) = \sup_{0 \neq a \in A_0 \cap A_1} \frac{\| a \|_A}{J(t^{-1}, a; A_0, A_1)}$$

$$= \sup_{\| a \|_{A_0} \leqslant 1,\ \| a \|_{A_1} \leqslant t} \| a \|_A,$$

$$\left(\nu_A(t) = \nu_A(t, A_0, A_1) = \sup_{0 \neq a \in A} \frac{tK(t^{-1}, a; A_0, A_1)}{\| a \|_A} \right).$$

We note that μ_A, $\nu_A \in \mathscr{P}$, $\mu_A(1)$ is the norm of imbedding $A_0 \cap A_1$ into A, and $\nu_A(1)$ is the norm of imbedding A into $A_0 + A_1$.

Let us investigate properties of these functions which we will need; other properties of μ_A in the case of symmetric spaces will be considered in Section 5.

Proposition 1. (a) *Suppose that $A_0 \subset A_1$. If A_0 is non-closed in A_1 then $\mu_{A_0}(t) = 1$ for all $t > 0$; if A_0 is closed in A_1 then $\mu_{A_0}(t) \approx \min(1, t)$*[1].

(b) *If $A_0 \cap A_1$ is a non-closed subspace in both A_0 and A_1 then $\mu_{A_0 \cap A_1}(t) = \max(1, t)$.*

(c) *If $A_0 \cap A_1$ is dense in both A_0 and A_1 and if A is intermediate space between A_0 and A_1 then $\nu_A(t, A_0, A_1) = \mu_{A^*}(t, A_1^*, A_0^*)$.*

Proof. Obviously $\min(1, t)\, \mu_{A_0}(1) \leqslant \mu_{A_0}(t) \leqslant 1$ for all $t > 0$.

(a) If A_0 is non-closed in A_1 then there exists a sequence $\{a_n\} \subset A_0$ such that $\| a_n \|_{A_0} = 1$ and $\| a_n \|_{A_1} \to 0$. Hence

$$\mu_{A_0}(t) \geqslant \lim_{n \to \infty} \frac{\| a_n \|_{A_0}}{J(t^{-1}, a_n)} = 1.$$

[1] The symbol $f(t) \approx g(t)$ means that there exist positive constants c_1, c_2 such that $c_1 f(t) \leqslant g(t) \leqslant c_2 f(t)$ for all $t > 0$.

If A_0 is closed in A_1 then $\|a\|_{A_0} \leqslant C_1 \|a\|_{A_1}$ for each $a \in A_0$ and so

$$\frac{\|a\|_{A_0}}{J(t^{-1}, a)} \leqslant \min(1, t\,\|a\|_{A_0}/\|a\|_{A_1}) \leqslant \max(1, C_1) \min(1, t).$$

(b) Since $\|a\|_{A_0 \cap A_1} \leqslant \max(1, t)\, J(t^{-1}, a)$ it follows that $\mu_{A_0 \cap A_1}(t) \leqslant \max(1, t)$. From the assumptions there exist sequences $\{a_n^i\} \subset A_0 \cap A_1$ such that $\|a_n^i\|_{A_0 \cap A_1} = 1$ and $\lim_{n \to \infty} \|a_n^i\|_{A_i} = 0$, $i = 0, 1$. Hence

$$\begin{aligned}\mu_{A_0 \cap A_1}(t) &= \max(\mu_{A_0}(t), \mu_{A_1}(t)) \\ &\geqslant \max\left(\lim_{n \to \infty} \frac{\|a_n^1\|_{A_0}}{J(t^{-1}, a_n^1)}, \lim_{n \to \infty} \frac{\|a_n^0\|_{A_1}}{J(t^{-1}, a_n^0)}\right) = \max(1, t).\end{aligned}$$

(c) If $A_0 \cap A_1$ is dense in both A_0 and A_1 then (A_0^*, A_1^*) is a Banach couple and if A is an intermediate space between A_0 and A_1 then A^* contains $A_0^* \cap A_1^*$. Moreover since $K(t^{-1}, a; A_0, A_1)$ and $J(t, a^*; A_0^*, A_1^*)$ are dual norms (cf. [2]), it follows that

$$\begin{aligned}\nu_A(t) &= \sup_{a \in A} \frac{t}{\|a\|_A} \sup_{a^* \in A_0^* \cap A_1^*} \frac{|a^*(a)|}{J(t, a^*; A_0^*, A_1^*)} \\ &= \sup_{a^* \in A_0^* \cap A_1^*} \frac{1}{J(t^{-1}, a^*; A_1^*, A_0^*)} \sup_{a \in A} \frac{|a^*(a)|}{\|a\|_A} \\ &= \sup_{a^* \in A_0^* \cap A_1^*} \frac{\|a^*\|_{A^*}}{J(t^{-1}, a^*; A_1^*, A_0^*)} = \mu_{A^*}(t.\, A_1^*, A_0^*).\end{aligned}$$

The following proposition is similar to Lemma 7. III in [1] and Lemma 4 in [6] (for completeness sake we give a proof).

PROPOSITION 2 (Necessary Conditions). *Let* $(A, B) \in \operatorname{Int}(\bar{A}, \bar{B})$.

(a) *If* $A \not\subset \bar{A}_i^{A_0 + A_1}$ *then* $B \supset B_{1-i}$, $i = 0, 1$.

(b) *If* $A_0 \cap A_1$ *is dense in both* A_0 *and* A_1 *then*

$$\mu_B(t, B_0, B_1)\, \mu_{A^*}(t, A_1^*, A_0^*) \leqslant Ct \tag{2}$$

for all $t \geqslant 0$.

Proof. (a) Let $a \in A$, $a \notin \bar{A}_i^{A_0 + A_1}$, and let f be a bounded linear functional on $A_0 + A_1$ vanishing on $\bar{A}_i^{A_0 + A_1}$ and $f(a) = 1$.

For any $b \in B_{1-i}$ the linear operator $Tx = f(x)\,b$ belongs to $L(\bar{A}, \bar{B})$. Hence $b = Ta \in B$ and

$$\|b\|_B = \|Ta\|_B \leqslant \|T\|_{A \to B} \|a\|_A \leqslant C \|f\|_{A_{1-i}^*} \|b\|_{B_{1-i}} \|a\|_A.$$

This proves assertion (a).

(b) We consider the one-dimensional operator $T\colon A_0 + A_1 \to B_0 \cap B_1$, $Ta = a^*(a)\,b$, where $a^* \in A_0^* \cap A_1^* = (A_0 + A_1)^*$ and $b \in B_0 \cap B_1$. We have

$$\|T\|_{A_i \to B_i} = \|b\|_{B_i} \sup_{\|a\|_{A_i} \leqslant 1} |a^*(a)| = \|b\|_{B_i} \|a^*\|_{A_i^*}, \qquad i = 0, 1,$$

and

$$\|T\|_{A \to B} = \|b\|_B \|a^*\|_{A^*}.$$

The interpolation property implies that there exists a constant $C > 0$ such that

$$\|b\|_B \|a^*\|_{A^*} \leqslant C \max\{\|b\|_{B_0} \|a^*\|_{A_0^*}, \|b\|_{B_1} \|a^*\|_{A_1^*}\},$$
$$\forall b \in B_0 \cap B_1, \forall a^* \in A_0^* \cap A_1^*. \qquad (3)$$

Since

$$\max\{\|b\|_{B_0} \|a^*\|_{A_0^*}, \|b\|_{B_1} \|a^*\|_{A_1^*}\} \leqslant tJ(t^{-1}, b; B_0, B_1)\, J(t^{-1}, a^*; A_1^*, A_0^*)$$

it follows from (3) by taking the supremum over all $b \in B_0 \cap B_1$ and all $a^* \in A_0^* \cap A_1^*$ that inequality (2) holds.

3. Results for Banach Spaces

From the definition we have $(A_0 + A_1, B_0 + B_1) \in \operatorname{Int}(\bar{A}_1, \bar{B})$. We will be interested in taking a smaller space B in the place of the sum $B_0 + B_1$. In certain cases the next theorem determines how large B must be whenever $(A_0 + A_1, B)$ belongs to $\operatorname{Int}(\bar{A}, \bar{B})$. The closure of $B_0 \cap B_1$ in B_i will be denoted by B_i^0, $i = 0, 1$.

Theorem 1. *Suppose that $A_0 \neq A_1$ and $(A_0 + A_1, B) \in \operatorname{Int}(\bar{A}, \bar{B})$.*

(a) *If $A_0 \cap A_1$ is not dense in both A_0 and A_1 then $B = B_0 + B_1$.*

(b) *If $A_0 \cap A_1$ is dense in A_0 and not dense in A_1 then $B \supset B_0^0 + B_1$.*

(c) *If $A_0 \cap A_1$ is dense in both A_0 and A_1 then $B \supset B_0^0 + B_1^0$.*

Proof. (a) We note that if $A_0 \cap A_1$ is not dense in A_{1-i}, or equivalently that A_i is not dense in $A_0 + A_1$, then $A_0 + A_1 \not\subset \bar{A}_i^{A_0+A_1}$ and by by Proposition 2(a) we have $B \supset B_{1-i}$ $(i = 0, 1)$. This means that $B = B_0 + B_1$.

(b) From the above we have $B \supset B_1$. Moreover, since $(A_0 + A_1, B \cap B_0) \in \mathrm{Int}((A_1, A_0 + A_1), (B_0 \cap B_1, B_0))$ it is sufficient to prove

$$\text{if } A_0 \subset A_1,\ A_0 \neq A_1,\ A_0 \text{ is dense in } A_1,\ B_0 \subset B_1 \text{ and } (A_1, B) \in \mathrm{Int}(\bar{A}, \bar{B}) \text{ then } B \supset B_1^0 \tag{4}$$

Before the proof of (4) we note that if A_0 is a proper and dense subspace of A_1 then $A_1^* \subseteq A_0^*$ and A_1^* is non-closed in A_0^*.

On the contrary, if A_1^* is closed in A_0^* then it is closed in the topology $\sigma(A_0^*, A_0)$. Since A_1^* is dense in A_0^* in the topology $\sigma(A_0^*, A_0)$ we have $A_0^* = A_1^*$ and density of A_0 in A_1 implies $A_0 = A_1$.

Now, we prove (4).

Applying Proposition 1(a), the fact that A_1^* is non-closed in A_0^*, and Proposition 2(b) we have

$$\mu_B(t, B_0, B_1) = \mu_B(t, B_0, B_1)\, \mu_{A_1^*}(t, A_1^*, A_0^*) \leqslant Ct$$

for all $t > 0$. Hence

$$\| b \|_B \leqslant C \max(t \| b \|_{B_0}, \| b \|_{B_1})$$

for each $b \in B_0$ and all $t > 0$. Taking $t \to 0^+$ we get

$$\| b \|_B \leqslant C \| b \|_{B_1}, \qquad \forall b \in B_1.$$

Density of B_0 in B_1^0 implies that the above inequality holds for each $b \in B_1^0$. Hence $B \supset B_1^0$.

(c) By the same arguments as in the proof of (b) we have that $B \supset B_0^0$ and $B \supset B_1^0$, and so $B \supset B_0^0 + B_1^0$.

From the above theorem it is easy to construct examples of non-interpolation spaces by first summing $A_0 + A_1$ and then making B less than $B_0^0 + B_1^0$.

In the proof of next theorem which gives non-interpolation spaces we need the following lemmas.

Lemma 1. *Suppose that*

$$\text{there exists a sequence } \{b_n\} \subset B_0 \cap B_1 \text{ such that } \| b_n \|_{B_0 \cap B_1} = 1,\ \| b_n \|_{B_1} \to 0 \text{ and } \| b_n \|_B \geqslant C_2 \text{ for some } C_2 > 0. \tag{5}$$

If $A \not\subset \tilde{A}_0$ then there exists a sequence $\{T_n\}$ of operators such that

$$\sup_n \| T_n \|_{L(\bar{A}, \bar{B})} \leqslant 1 \qquad \text{and} \qquad \limsup_{n \to \infty} \| T_n \|_{A \to B} = \infty. \tag{6}$$

Proof (cf. [12]). Let $t_n^{-1} = \| b_n \|_{B_1}$. Consider the linear operators $T_n x = b_n f_n(x)$, where f_n are bounded liner functionals on $A_0 + A_1$ with

$$| f_n(x)| \leqslant K(t_n, x; A_0, A_1) \qquad \text{and} \qquad f_n(a) = K(t_n, a; A_0, A_1)$$

and $a \in A$, $\| a \|_A \leqslant 1$, $a \notin \tilde{A}_0$. The existence of such functionals follows from the Hahn–Banach theorem.

If $x \in A_i$, then by (5)

$$\begin{aligned} \| T_n x \|_{B_i} &= \| b_n \|_{B_i} \, | f_n(x)| \leqslant \| b_n \|_{B_i} K(t_n, x; A_0, A_1) \\ &\leqslant \| b_n \|_{B_i} t_n^i \, \| x \|_{A_i} = \| x \|_{A_i}, \qquad i = 0, 1. \end{aligned}$$

and

$$\begin{aligned} &\lim_{n \to \infty} \sup \| T_n a \|_B \\ &\quad = \limsup_{n \to \infty} \| b_n \|_B \, | f_n(a)| \\ &\quad = \lim_{n \to \infty} \sup \| b_n \|_B \, K(t_n, a; A_0, A_1) \\ &\quad \geqslant C_2 \lim_{n \to \infty} \sup K(t_n, a; A_0, A_1) \\ &\quad = C_2 \lim_{t \to \infty} K(t, a; A_0, A_1) = \infty, \end{aligned}$$

and the proof is complete,

LEMMA 2. *If $A_1 \subset \bar{A}_{\varphi, \infty}$ and* $\lim_{t \to \infty} \varphi(t)/t = 0$ *then $A_1 \subset A_0$.*

Proof. First method. From assumptions there exists $C_3 > 0$ such that

$$K(t, a; A_0, A_1) \leqslant C_3 \varphi(t) \| a \|_{A_1}, \qquad \forall a \in A_1, \forall t > 0, \tag{7}$$

and there exists $t_0 > 0$ such that $\varphi(t_0)/t_0 \leqslant 1/(4C_3)$. By the definition of the K-functional we can write (for t_0 fixed) $a = a_1 + b_1$ with

$$\begin{aligned} &\| a_1 \|_{A_0} + t_0 \, \| b_1 \|_{A_1} \\ &\quad \leqslant 2K(t_0, a; A_0, A_1) \text{ [from assumption (7)]} \\ &\quad \leqslant 2C_3 \varphi(t_0) \| a \|_{A_1} \leqslant \frac{t_0}{2} \, \| a \|_{A_1}, \end{aligned}$$

i.e.,

$$\|a_1\|_{A_0} \leqslant \frac{t_0}{2} \|a\|_{A_1} \quad \text{and} \quad \|b_1\|_{A_1} \leqslant 2^{-1} \|a\|_{A_1}.$$

Then similarly $b_1 = a_2 + b_2$ where

$$\|a_2\|_{A_0} \leqslant \frac{t_0}{2} \|b_1\|_{A_1} \leqslant \frac{t_0}{4} \|a\|_{A_1} \quad \text{and}$$

$$\|b_2\|_{A_1} \leqslant 2^{-1} \|b_1\|_{A_1} \leqslant 2^{-2} \|a\|_{A_1}.$$

Proceeding by induction we get $a = (a_1 + a_2 + \cdots + a_n) + b_n$ where

$$\|a_n\|_{A_0} \leqslant 2^{-n} t_0 \|a\|_{A_1} \quad \text{and} \quad \|b_n\|_{A_1} \leqslant 2^{-n} \|a\|_{A_1}.$$

Since $b_n \to 0$ in A_1 and $\sum_1^\infty a_n \in A_0$ we get $a \in A_0$ and so $A_1 \subset A_0$.

Second method (if additionally $\bar{A}_{\varphi, \infty} = A_0 + A_1$). First, we note that if $a \in A_0^0 + A_1^0$ then $K(t, a; A_0^0, A_1^0) = K(t, a; A_0, A_1)$ and so

$$(A_0^0, A_1^0)_{\varphi, \infty} = (A_0, A_1)_{\varphi, \infty} \cap (A_0^0 + A_1^0).$$

Hence, if $\bar{A}_{\varphi, \infty} = A_0 + A_1$, then $(A_0^0, A_1^0)_{\varphi, \infty} = A_0^0 + A_1^0$ and we may assume that $A_0 \cap A_1$ is dense in both A_0 and A_1. From the fact $(A_0 + A_1)^* = A_0^* \cap A_1^*$ and from Proposition 1(c) (under the above density assumption) we have

$$\frac{\varphi(t)}{t} \geqslant v_{\bar{A}_{\varphi,\infty}}(t^{-1}, A_0, A_1) \approx v_{A_0 + A_1}(t^{-1}, A_0, A_1)$$

$$= \mu_{(A_0 + A_1)^*}(t^{-1}, A_1^*, A_0^*) = \mu_{A_0^* \cap A_1^*}(t^{-1}, A_1^*, A_0^*).$$

There are three mutually exclusive possibilities for A_0 and A_1: (i) $A_1 \subset A_0$, (ii) $A_0 \subset A_1$ and $A_0 \neq A_1$, (iii) $A_0 \cap A_1 \neq A_0$ and $A_0 \cap A_1 \neq A_1$.

Assumption $\lim_{t \to \infty} \varphi(t)/t = 0$ and Proposition 1(a), 1(b) give that the second and third cases are impossible. Hence $A_1 \subset A_0$.

From the equality $\bar{A}_{0, \infty} = \tilde{A}_0$ and Lemma 2 immediately follows the Aronszajn–Gagliardo result (see [1]; see also [9, 12, 14]): if $\tilde{A}_0 = A_0 + A_1$ then $A_0 = A_0 + A_1$, i.e., $A_1 \subset A_0$.

THEOREM 2. *If $A_i \neq A_0 + A_1$ and $B_0 \cap B_1$ is a non-closed subspace of B_{1-i} then $(A_{1-i}, B_i) \notin \mathrm{Int}(\bar{A}, \bar{B})$ ($i = 0$ or 1).*

Proof for $i = 0$. From the assumption there exists a sequence $\{b_n\} \subset B_0 \cap B_1$ such that $\|b_n\|_{B_0 \cap B_1} = 1$ and $\|b_n\|_{B_1} \to 0$. It follows that

$$\|b_n\|_{B_0 \cap B_1} = 1, \qquad \|b_n\|_{B_1} \to 0, \qquad \|b_n\|_{B_0} = 1.$$

Since $A_0 \neq A_0 + A_1$ we have by Lemma 2 that $A_1 \not\subset \tilde{A}_0$. Applying Lemma 1 to couples $\bar{A}$, $\bar{B}$ and spaces $A = A_1$, $B = B_0$ we have a sequence $\{T_n\}$ of operators such that

$$\sup_n \| T_n \|_{L(\bar{A},\bar{B})} \leqslant 1 \qquad \text{and} \qquad \limsup_{n\to\infty} \| T_n \|_{A_1 \to B_0} = \infty.$$

Hence $(A_1, B_0) \notin \operatorname{Int}(\bar{A}, \bar{B})$.

Aronszajn and Gagliardo in [1] investigated when A_0 or A_1 belongs to the set $\operatorname{Int}(A_0 + A_1, A_0 \cap A_1)$, i.e., when A_0 or A_1 is an interpolation space between sum $A_0 + A_1$ and intersection $A_0 \cap A_1$ (see also [12]). Now, we consider the problem when A_0 or A_1 belongs to a bigger set $\operatorname{Int}((A_0, A_1), (A_1, A_0))$.

THEOREM 3. *Let* $\bar{A} = (A_0, A_1)$, $\bar{B} = (A_1, A_0)$ *and suppose that* $A_0 \neq A_0 \cap A_1$, $A_1 \neq A_0 \cap A_1$.

(a) *If* $A_0 \cap A_1$ *is a non-closed subspace in* A_i, *then* $A_{1-i} \notin \operatorname{Int}(\bar{A}, \bar{B})$ ($i = 0$ *or* 1).

(b) *If* $A_0 \cap A_1$ *is closed in* A_0 *but not in* A_1, *then* $A_1 \in \operatorname{Int}(\bar{A}, \bar{B})$ *if and only if* $A_0 \cap A_1$ *is dense in* A_1.

(c) *If* $A_0 \cap A_1$ *is closed in both* A_0 *and* A_1, *then* $A_0, A_1 \notin \operatorname{Int}(\bar{A}, \bar{B})$.

Proof. (a) This is a particular case of Theorem 2.

(b) If $A_0 \cap A_1$ is dense in A_1 then we have

$$(A_0 + A_1)^0 = A_0^0 + A_1^0 = (A_0 \cap A_1) + A_1^0 = A_1.$$

Hence, if $T \in L(\bar{A}, \bar{B})$ then T is bounded from $(A_0 + A_1)^0 = A_1$ into itself. On the other hand, if $A_1 \in \operatorname{Int}(\bar{A}, \bar{B})$ then $A_1 \subset \bar{A}_0^{A_0+A_1}$ (if $A_1 \not\subset \bar{A}_0^{A_0+A_1}$ then by Proposition 2(a) we get $A_1 \supset A_0$). Hence

$$A_1 \subset \bar{A}_0^{A_0+A_1} \cap \bar{A}_1^{A_0+A_1} = (A_0 + A_1)^0 = A_0^0 + A_1^0 = A_1^0,$$

i.e., $A_1 = A_1^0$.

(c) Let $A_0 \cap A_1$ be closed in both A_0, A_1 and let $A \in \operatorname{Int}(\bar{A}, \bar{B})$. Since $\bar{A}_i^{A_0+A_1} = A_i$ ($i = 0, 1$) we have four mutually exclusive possibilities for A: (i) $A \subset A_0$ and $A \subset A_1$, (ii) $A \subset A_0$ and $A \not\subset A_1$, (iii) $A \not\subset A_0$ and $A \subset A_1$, (iv) $A \not\subset A_0$ and $A \not\subset A_1$.

The first case gives $A = A_0 \cap A_1$. Proposition 2(a) implies that the second and third cases are impossible, and the fourth case has the form $A \supset A_0$ and $A \supset A_1$, i.e., $A = A_0 + A_1$. Hence, only $A_0 \cap A_1$ and $A_0 + A_1$ are interpolation spaces with respect to (A_0, A_1) and (A_1, A_0).

4. Results for Symmetric Spaces

The necessary condition in Proposition 2(b) required the assumption of density of the intersection in each of the spaces. In the case of symmetric spaces (or even Banach lattices of measurable functions) it is possible to obtain a necessary condition for interpolation by taking associated spaces in the place of conjugate spaces.

A Banach space E of equivalence classes of measurable functions on $I=(0, l)$, $0<l\leqslant\infty$, is said to be a *symmetric space* (on I) if $y\in E$ and measurable x are such that $x^*(t)\leqslant y^*(t)$ for $t\in I$, then $x\in E$ and $\|x\|E\leqslant \|y\|_E$ (cf. [9]). Here x^* denotes the non-increasing rearrangement of $|x|$.

The associate space E' of a symmetric space E is the collection of all measurable functions x for which

$$\|x\|_{E'}=\sup_{\|y\|_E\leqslant 1}\int_I |x(t)\,y(t)|\,dt<\infty.$$

The fundamental function $\varphi=\varphi_E$ of a symmetric space E on I is defined for $t\in I$ as $\varphi_E(t)=\|1_{(0,t)}\|_E$, where $1_{(0,t)}$ is the characteristic function of the interval $(0, t)$.

First we describe a necessary condition for the interpolation of symmetric spaces. Namely, if $(E, F)\in \operatorname{Int}(\bar{E}, \bar{F})$ where $\bar{E}=(E_0, E_1)$ and $\bar{F}=(F_0, F_1)$, then

$$\mu_F(t, F_0, F_1)\,\mu_{E'}(t, E_1', E_0')\leqslant Ct, \qquad \forall t>0. \tag{2'}$$

For the proof we consider the one-dimensional operator $T\colon E_0+E_1\to F_0\cap F_1$ defined by

$$Tx(t)=b(t)\int_I x(s)\,a(s)\,ds, \qquad b\in F_0\cap F_1,\ a\in E_0'\cap E_1'.$$

Then

$$\|T\|_{E_i\to F_i}=\|b\|_{F_i}\sup_{\|x\|_{E_i}\leqslant 1}\left|\int_I x(s)\,a(s)\,ds\right|$$
$$=\|b\|_{F_i}\,\|a\|_{E_i'}, \qquad i=0, 1,$$

and $\|T\|_{E\to F}=\|b\|_F\,\|a\|_{E'}$.

The interpolation property implies that there exists a positive constant C such that

$$\|b\|_F\,\|a\|_{E'}\leqslant C\max\{\|b\|_{F_0}\,\|a\|_{E_0'}\,\|b\|_{F_1}\,\|a\|_{E_1'}\},$$
$$\forall b\in F_0\cap F_1,\ \forall a\in E_0'\cap E_1'. \tag{3'}$$

It can be proved that inequality (3′) is equivalent to (2′).

Now, we prove that condition (3′) or the equivalent condition (2′) gives more information than the well-known earlier (see [12, 13]) necessary condition for interpolation, i.e., condition (3′) with $a = 1_{(0,t)}$ and $b = 1_{(0,s)}$.

Let E_0, E_1, and E be symmetric spaces on I with the fundamental functions φ_0, φ_1, and φ, respectively.

THEOREM 4. *Let $I=(0,1)$. Suppose that $L_\infty \subset E \subset E_1$ and that either E coincides with E'' or L_∞ is dense in E_1. If $E \in \operatorname{Int}(L_\infty, E_1)$ then one of the three conditions holds*:

$$E = L_\infty \qquad or \qquad E = E_1 \qquad or \qquad \liminf_{t\to 0^+} \frac{\varphi(t)}{\varphi_1(t)} = \infty.$$

Proof. Let $\varphi(0^+) := \lim_{t\to 0^+} \varphi(t) = 0$; in the opposite case $E = L_\infty$. From (3′) with $a = 1_{(0,t)}$we have

$$\|b\|_E \leqslant C \max\left\{\varphi(t)\|b\|_{L_\infty}, \frac{\varphi(t)}{\varphi_1(t)}\|b\|_{E_1}\right\}$$

for any $b \in L_\infty$ and all $t \in I$. If $\liminf_{t\to 0^+} (\varphi(t)/\varphi_1(t)) = C_4$ then from the above

$$\|b\|_E \leqslant CC_4\|b\|_{E_1}, \qquad \forall b \in L_\infty. \tag{8}$$

First, let $E = E''$. If $x \in E_1$ then there exists a sequence (x_n) of bounded functions such that $0 \leqslant x_n \nearrow |x|$. Since $\|x_n\|_E \leqslant CC_4\|x_n\|_{E_1} \leqslant CC_4\|x\|_{E_1}$ we have by the Fatou property of E that $x \in E$ and $\|x\|_E = \lim_{n\to\infty} \|x_n\|_E \leqslant CC_4 \lim_{n\to\infty} \|x_n\|_{E_1} = CC_4\|x\|_{E_1}$. Hence $E = E_1$.

Second, if L_∞ is dense in E_1 then inequality (8) holds for any $b \in E_1$. Hence $E = E_1$.

COROLLARY 1. *Let $I=(0,1)$. If $1 \leqslant q < p < \infty$ then $L_\infty \subset L_{pq} \subset L_p$ and $L_{pq} \notin \operatorname{Int}(L_\infty, L_p)$ (see [10, Ex. 1]). More generally, if $1 < p < \infty$ and $1 \leqslant q < r \leqslant \infty$ then $L_\infty \subset L_{pq} \subset L_{pr}$ and $L_{pq} \notin \operatorname{Int}(L_\infty, L_{pr})$.*

Finally, using (3′) the following theorem can be proved in the same way as Theorem 4.

THEOREM 5. *Let E_0, E_1, and E be symmetric spaces on I such that $E_0 \subset E \subset E_1$, $E \neq E_1$, and either E coincides with E'' or E_0 is dense in E_1. If $\varphi(t) = \varphi_1(t)$ for $t \in I$ and $\liminf_{t\to 0^+} (\varphi(t)/\varphi_0(t)) = 0$ then $E \notin \operatorname{Int} \bar{E}$.*

COROLLARY 2. *We consider the Lorentz spaces L_{pq}, L_{pr}, and L_{st} on $I=(0,1)$. If $1 \leqslant q < r$ and $1 \leqslant p < s$ then $L_{st} \subset L_{pq} \subset L_{pr}$ and $L_{pq} \notin \operatorname{Int}(L_{st}, L_{pr})$.*

Remark 1. In Theorem 4 (and Theorem 5) assumptions that E coincides with E'' or L_∞ is dense in E_1 are important. Namely, if E_1 is non-separable, different from L_∞, and takes for E the closure of L_∞ in E_1, then $E \in \operatorname{Int}(L_\infty, E_1)$ and none of the three conditions in the assertion of Theorem 4 is satisfied.

5. Fundamental Function μ for Symmetric Spaces

Let E_0, E_1, and E be symmetric spaces on I with the fundamental functions φ_0, φ_1, and φ, respectively. Put $\varphi_{10}(t) = \varphi_1(t)/\varphi_0(t)$.

1. If $E \in \operatorname{Int} \bar{E}$ then taking $a = 1_{(0,t)} \in E_0 \cap E_1$ in the definition of μ_E and in (3′) we obtain

$$\frac{\varphi(t)}{\varphi_0(t)} \leqslant \mu_E(\varphi_{10}(t), E_0, E_1) \leqslant C \frac{\varphi(t)}{\varphi_0(t)}, \qquad \forall t \in I. \tag{9}$$

If, moreover, $I = \mathbb{R}_+$, $C = 1$, and $\varphi_{10}(\mathbb{R}_+) = \mathbb{R}_+$ then

$$\begin{aligned} \mu_E(t) &= \sup_{s>0} \mu_E(s) \min(1, t/s) \\ &= \sup_{s>0} \mu_E(\varphi_{10}(s)) \min(1, t/\varphi_{10}(s)) \\ &= \sup_{s>0} \frac{\varphi(s)}{\varphi_0(s)} \min(1, t/\varphi_{10}(s)) \\ &= \sup_{s>0} \varphi(s) \min\left(\frac{1}{\varphi_0(s)}, \frac{t}{\varphi_1(s)}\right), \end{aligned}$$

i.e.,

$$\mu_E(t, E_0, E_1) = \sup_{s>0} \varphi(s) \min\left(\frac{1}{\varphi_0(s)}, \frac{t}{\varphi_1(s)}\right). \tag{10}$$

In a particular case, $\mu_E(t, L_\infty, L_1) = \varphi(t)$. Assumption $\varphi_{10}(\mathbb{R}_+) = \mathbb{R}_+$ is essential in formula (10). Namely, if $\varphi_0 = \varphi_1 = \varphi$ then the right-hand side in equality (10) is equal to $\min(1, t)$ while the left-hand side can be equal to 1 as it was in Proposition 1(a).

In particular, if $1 \leqslant p_0 < p < p_1 \leqslant \infty$ then from the M. Riesz interpolation theorem and the above

$$\mu_{L_p}(t, L_{p_0}, L_{p_1}) = t^{(1/p_0 - 1/p)/(1/p_0 - 1/p_1)}. \tag{10′}$$

2. Assumption $E \in \operatorname{Int} \bar{E}$ is essential in formula (10). Namely, if $I=(0, \infty)$ and $1<p<\infty$ then

$$\begin{aligned}
&\mu_{\Lambda(L_p+L_\infty)}(t, L_{p\infty}, L_\infty) \\
&\quad \approx \|\min(s^{-1/p}, t)\|_{\Lambda(L_p+L_\infty)} \\
&\quad = \int_0^\infty \min(s^{-1/p}, t)\, ds \min(s^{1/p}, 1) = \int_0^1 \min(s^{-1/p}, t)\, ds^{1/p} \\
&\quad = \begin{cases} t & \text{if } 0<t \leqslant 1, \\ 1+\ln t & \text{if } t \geqslant 1. \end{cases}
\end{aligned}$$

However the right-hand side in equality (10) is equal to $\min(1, t)$.

3. For E_0, E_1, and E on $I=(0, \infty)$ such that $E_0 \cap E_1 \subset E$ let the following inequalities hold,

$$\left\| \frac{1_{(0,t)}}{\varphi_1} \right\|_E \leqslant C_1 \frac{\varphi(t)}{\varphi_1(t)} \quad \text{and} \quad \left\| \frac{1_{(t,\infty)}}{\varphi_0} \right\|_E \leqslant C_0 \frac{\varphi(t)}{\varphi_0(t)} \tag{11}$$

for some $C_0, C_1>0$ and all $t>0$. Then

$$\frac{\varphi(t)}{\varphi_0(t)} \leqslant \mu_E(\varphi_{10}(t), E_0, E_1) \leqslant (C_0+C_1) \frac{\varphi(t)}{\varphi_0(t)}. \tag{12}$$

Namely, for any $a \in E_0 \cap E_1$ such that $\|a\|_{E_0} \leqslant 1$ and $\|a\|_{E_1} \leqslant \varphi_{10}(t)$ we have $a^*(s) \leqslant 1/\varphi_0(s)$ and $a^*(s) \leqslant \varphi_{10}(t)/\varphi_1(s)$ a.e., and so

$$\begin{aligned}
\|a\|_E &= \|a^*\|_E \leqslant \|a^* 1_{(0,t)}\|_E + \|a^{*1}(t, \infty)\|_E \\
&\leqslant \varphi_{10}(t) \left\| \frac{1_{(0,t)}}{\varphi_1} \right\|_E + \left\| \frac{1_{(t,\infty)}}{\varphi_0} \right\|_E \\
&\qquad\qquad \text{[from assumption (11)]} \\
&\leqslant C_1 \varphi_{10}(t) \frac{\varphi(t)}{\varphi_1(t)} + C_0 \frac{\varphi(t)}{\varphi_0(t)} = (C_0+C_1) \frac{\varphi(t)}{\varphi_0(t)},
\end{aligned}$$

i.e.,

$$\mu_E(\varphi_{10}(t)) \leqslant (C_0+C_1) \frac{\varphi(t)}{\varphi_0(t)}.$$

Assumptions of type (11) can be found in [11], where the K-functional for symmetric spaces is computed.

For example, if $t^a \varphi(t)/\varphi_0(t)$ is a decreasing function for some $a>0$ then

$$\frac{\varphi(t)}{\varphi_0(t)} = \mu_E(1/\varphi_0(t), E_0, L_\infty) \leqslant 2(2+1/a) \frac{\varphi(t)}{\varphi_0(t)}. \tag{12'}$$

It is sufficient to prove inequalities (11). The first inequality with $C_1 = 1$ is obvious; proof of the second inequality is the following (cf. [11]):

$$2^{-1}\left\|\frac{1_{(t,\infty)}}{\varphi_0}\right\|_E \leqslant \left\|\frac{1_{(t,\infty)}}{\varphi_0}\right\|_{\Lambda(E)} = \int_0^\infty \left(\frac{1_{(t,\infty)}}{\varphi_0}\right)^* (s)\, d\varphi(s)$$

$$= \int_0^\infty \frac{d\varphi(s)}{\varphi_0(s+t)} \leqslant \int_0^t \frac{d\varphi(s)}{\varphi_0(t)} + \int_t^\infty \frac{d\varphi(s)}{\varphi_0(s)}$$

$$\leqslant \frac{\varphi(t)}{\varphi_0(t)} + \int_t^\infty \frac{\varphi(s)}{\varphi_0(s)} \frac{ds}{s} \leqslant \frac{\varphi(t)}{\varphi_0(t)} + \frac{\varphi(t)\, t^a}{\varphi_0(t)} \int_t^\infty \frac{ds}{s^{1+a}}$$

$$= (1 + 1/a)\frac{\varphi(t)}{\varphi_0(t)}.$$

Inequalities (12′) can also be obtained from the formula $\mu_E(1/\varphi_0(t), E_0, L_\infty) \approx \sup_{\|a\|_{E_0} \leqslant 1} \|a^* 1_{(t,\infty)}\|_E$ which was proved in [3, Th. 7] in connection with the Nikolski type inequality.

4. If $I = (0, 1)$ and $1 < p < \infty$, $1 \leqslant q < \infty$, then

$$\mu_{L_{pq}}(t, L_{p\infty}, L_\infty) \approx \|\min(s^{-1/p}, t)\|_{L_{pq}}$$

$$\approx \left(\frac{q}{p}\int_0^1 [s^{1/p}\min(s^{-1/p}, t)]^q \frac{ds}{s}\right)^{1/q}$$

$$= \begin{cases} t & \text{if } 0 < t \leqslant 1, \\ (1 + q\ln t)^{1/q} & \text{if } t \geqslant 1. \end{cases}$$

It would be of interest to compute $\mu_{L_{pq}}(t, L_p, L_\infty)$.

COROLLARY 3. *Let* $I = (0, 1)$. *If* $1 < p < \infty$ *and* $1 < q \leqslant \infty$ *then* $L_{p1} \subset L_{pq} \subset L_1$ *and* $L_{pq} \notin \mathrm{Int}(L_{p1}, L_1)$.

Proof. Suppose that $L_{pq} \in \mathrm{Int}(L_{p1}, L_1)$. Then by (2′) the function

$$f(t) = \mu_{L_{pq}}(t, L_{p1}, L_1)\, \mu_{L_{p'q'}}(t, L_\infty, L_{p'\infty})/t$$

is bounded. However, if $0 < t \leqslant 1$ then by (9) we have $1 \leqslant \mu_{L_{pq}}(t, L_{p1}, L_1) \leqslant C$ and by the above $\mu_{L_{p'q'}}(t, L_\infty, L_{p'\infty}) = t\mu_{L_{p'q'}}(t^{-1}, L_{p'\infty}, L_\infty) \approx t(1 + q' \ln \frac{1}{t})^{1/q'}$. Hence $\lim_{t\to 0^+} f(t) = \infty$, i.e., f is unbounded and we have a contradiction.

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